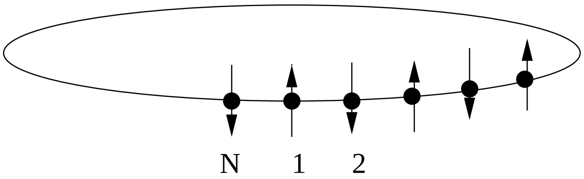
Lecture VIII: Quantum Antiferromagnetism

We have seen that the quantum Heisenberg Ferromagnetic spin chain is characterised by a magnetically ordered ground state with free particle-like elementary spin wave excitations — magnons. What happens in the antiferromagnetic system?

▷ Antiferromagnet Heisenberg spin S chain

$$\hat{H} = J \sum_{m=1}^{N} \hat{\mathbf{S}}_m \cdot \hat{\mathbf{S}}_{m+1}, \qquad J > 0, \qquad \text{p.b.c. } \hat{\mathbf{S}}_{m+N} = \hat{\mathbf{S}}_m$$



Classical ground state (Néel) no longer an eigenstate
— nevertheless, it serves as reference for spin-wave expansion

In this case, it is convenient to implement transformation in which spins on one sublattice, say B, are rotated through 180^o about the x-axis,

$$\text{i.e.} \qquad \hat{S}_B^x \longmapsto \hat{S}_B^x, \qquad \hat{S}_B^y \longmapsto -\hat{S}_B^y, \qquad \hat{S}_B^z \longmapsto -\hat{S}_B^z$$

Note that transformation is said to be canonical:
i.e. it respects the canonical commutation relations

$$\hat{H} = -J \sum_m \left[\hat{S}_m^z \hat{S}_{m+1}^z - \frac{1}{2}(\hat{S}_m^+ \hat{S}_{m+1}^+ + \hat{S}_m^- \hat{S}_{m+1}^-) \right]$$

In rotated frame, classical ground state is ferromagnetic
but $\hat{S}_m^- \hat{S}_{m+1}^- \rightsquigarrow$ zero-point fluctuations (ZPF)

Applying spin wave approximation: $\hat{S}_m^z = S - a_m^\dagger a_m$, $\hat{S}_m^- \simeq (2S)^{1/2} a_m^\dagger$, etc.

$$\hat{H} = -NJS^2 + JS \sum_m \left[2a_m^\dagger a_m + a_m a_{m+1} + a_m^\dagger a_{m+1}^\dagger \right] + O(S^0)$$

$\rightsquigarrow$ processes that do not conserve particle number! (ZPF)

Turning to Fourier representation: $a_m = \frac{1}{N^{1/2}} \sum_k e^{-ikm} a_k$, etc., and using

$$\sum_{m=1}^{N} a_m a_{m+1} = \sum_{kk'} \overbrace{\frac{1}{N} \sum_{m=1}^{N} e^{-i(k+k')m}}^{\delta_{k+k',0}} e^{-ik} = \sum_k a_k a_{-k} e^{-ik} \equiv \sum_k a_k a_{-k} \overbrace{\frac{1}{2}(e^{ik} + e^{-ik})}^{\gamma_k = \cos k}$$

$$\hat{H} = -NJS(S+1) + JS\sum_k \begin{pmatrix} a_k^\dagger & a_{-k}\end{pmatrix}\begin{pmatrix}1 & \gamma_k\\ \gamma_k & 1\end{pmatrix}\begin{pmatrix}a_k\\ a_{-k}^\dagger\end{pmatrix} + O(S^0)$$

To diagonalise $\hat{H}$, might think of making unitary transformation. However, transformation must preserve canonical commutation relations. Achieved by (exercis, PS II)

▷ Bogoliubov transformation: (cf. Lorentz boost — preserves metric $g = \sigma_z$)

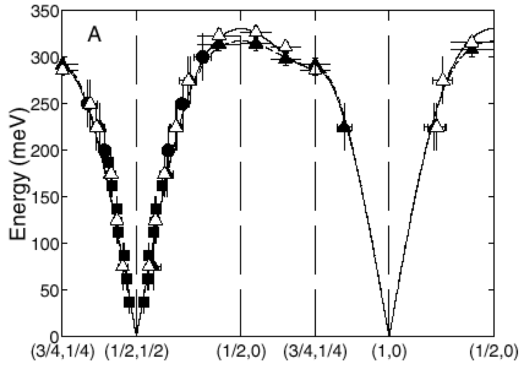
$$\begin{pmatrix}a_k\\ a_{-k}^\dagger\end{pmatrix} = \begin{pmatrix}\cosh\theta_k & -\sinh\theta_k\\ -\sinh\theta_k & \cosh\theta_k\end{pmatrix}\begin{pmatrix}\alpha_k\\ \alpha_{-k}^\dagger\end{pmatrix}$$

Off-diagonal terms removed by setting $\tanh(2\theta_k) = \gamma_k$

$$\begin{aligned}\hat{H} &= -NJS(S+1) + JS\sum_k |\sin k|\left(\alpha_k^\dagger\alpha_k + \alpha_{-k}\alpha_{-k}^\dagger\right) + O(S^0)\\ &= -NJS(S+1) + 2JS\sum_k |\sin k|\left[\alpha_k^\dagger\alpha_k + \frac{1}{2}\right] + O(S^0)\end{aligned}$$

$\rightsquigarrow$ linear (cf. relativistic) excitation spectrum (cf. phonons, photons, etc.)

Experiment?



▷ Average Magnetisation

- Do thermal fluctuations destroy magnetic order in ferromagnet?

$$\langle M\rangle = \langle \text{g.s.}|\frac{1}{N}\sum_i \hat{S}_i^z|\text{g.s.}\rangle = S - \langle \text{g.s.}|\frac{1}{N}\sum_i a_i^\dagger a_i|\text{g.s.}\rangle = S - \langle \text{g.s.}|\frac{1}{N}\sum_k a_k^\dagger a_k|\text{g.s.}\rangle$$

$$= S - \int \frac{d^dk}{(2\pi)^d}\overbrace{\frac{1}{e^{\omega_k/k_\mathrm{B}T}-1}}^{n_\mathrm{B}(T)} \overset{k_\mathrm{B}T\gg JS}{\simeq} S - \frac{k_\mathrm{B}T}{JS}\int_0^{1/a} k^{d-1}dk\frac{1}{k^2}$$

divergent for $T \neq 0$ in $d \leq 2$

i.e. In $d \leq 2$ long-range order destroyed by thermal fluctuations at any non-zero temperature (example of general principle: Mermin-Wagner theorem)

- At $T = 0$, do ZPF destroy long-range order in antiferromagnet?

  Referring to sublattice magnetisation (cf. $\langle m \rangle$ in rotated frame)

$$\begin{aligned}\langle M_{\text{s.l.}} \rangle &= \langle \text{g.s.}|\frac{1}{N}\sum_i \hat{S}_i^z|\text{g.s.}\rangle = S - \langle \text{g.s.}|\frac{1}{N}\sum_k a_k^\dagger a_k|\text{g.s.}\rangle = S - \frac{1}{N}\sum_k \sinh^2\theta_k \\ &= S - \int \frac{d^dk}{(2\pi)^d}\frac{1}{2}\left[(1-\gamma_k^2)^{-1/2} - 1\right] \sim \int_0^{1/a} k^{d-1} dk \frac{1}{k}\end{aligned}$$

  diverges in $d = 1$!

  i.e. quantum fluctuations destroy antiferromagnetic order in $d = 1$ even at $T = 0$!
  spin liquid phase

▷ FRUSTRATION

AF exchange interaction on "bipartite" lattice ⇝ Néel ordering
which, in $d > 1$, survives quantum ZPF

For non-bipartite lattice (such as triangular),
AF exchange interaction is said to be frustrated...

Can ZPF lead to spin liquid in higher dimensions...? subject of current research!